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Hal 80 – 82

3. Tunjukkan jika :

b). $z = \ln(x^2+y^2)$; maka $x z_x + y z_y = 1$

Jawab :

$$Z_x = \frac{\partial z}{\partial x} = 2x \cdot \frac{1}{x^2+y^2} = \frac{2x}{x^2+y^2}$$

$$Z_y = \frac{\partial z}{\partial y} = 2y \cdot \frac{1}{x^2+y^2} = \frac{2y}{x^2+y^2}$$

$$\Rightarrow x z_x + y z_y = x\left[\frac{2x}{x^2+y^2}\right] + y\left[\frac{2y}{x^2+y^2}\right]$$

$$= \frac{2x^2+2y^2}{x^2+y^2}$$

$$= \frac{2(x^2+y^2)}{x^2+y^2}$$

$$= 2$$

* Tidak terbukti bahwa $x z_x + y z_y = 1$

4. Dapatkan Diferensial Total dan d^2z dari

f). $z = \ln(x^2+y^2)$

▶ Diferensial Total

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= \frac{2x}{(x^2+y^2)}dx + \frac{2y}{(x^2+y^2)}dy$$

▶ d^2z

$$d^2z = f_{xx}(x,y)\,dx^2 + f_{xy}(x,y)\,dx\,dy + f_{yx}(x,y)\,dy\,dx + f_{yy}(x,y)\,dy^2$$

$$= f_{xx}(x,y)\,dx^2 + 2f_{xy}(x,y)\,dx\,dy + f_{yy}(x,y)\,dy^2$$

$$= Z_{xx}\,dx^2 + 2Z_{xy}\,dx\,dy + Z_{yy}\,dy^2$$

$$\Rightarrow Z_x = \frac{\partial z}{\partial x} = \frac{2x}{(x^2+y^2)}$$

$$\Rightarrow Z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \frac{2x^2-4x^2+2y^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\Rightarrow Z_y = \frac{\partial z}{\partial y} = \frac{2y}{(x^2+y^2)}$$

$$\Rightarrow Z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\Rightarrow Z_{xy} = Z_{yx} = \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial x} = \frac{(0).(x^2+y^2)-2y(2x)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\Rightarrow d^2z = Z_{xx}\,dx^2 + 2Z_{xy}\,dx\,dy + Z_{yy}\,dy^2$$

$$= \frac{2y^2-2x^2}{(x^2+y^2)^2}dx^2 + 2\left[\frac{-4xy}{(x^2+y^2)^2}\right]dx\,dy + \frac{2x^2-2y^2}{(x^2+y^2)^2}dy^2$$

$$= \frac{2y^2-2x^2}{(x^2+y^2)^2}dx^2 - \frac{8xy}{(x^2+y^2)^2}dx\,dy + \frac{2x^2-2y^2}{(x^2+y^2)^2}dy^2$$

5 Dapatkan nilai hampiran untuk :

e). Ralat relatif maksimum pada g jika g dihitung dari formula $s = \frac{1}{2} g t^2$ bila ada kemungkinan ralat 1% dalam pengukuran s dan ralat ¼ % dalam pengukuran t.

Diketahui = $s = \frac{1}{2} g t^2$ $\quad s_0 = s$ $\quad \Delta s = \frac{1}{100} s$

$t_0 = t$ $\quad \Delta t = \frac{1}{400} t$

$g = \frac{2s}{t^2}$

$G(s,t) = \frac{2s}{t^2}$

$$dG = \frac{\partial G}{\partial s} ds + \frac{\partial G}{\partial t} dt$$

$$= \frac{2}{t^2} ds + \frac{-4s}{t^3} dt$$

$$\Delta G = \frac{2}{t^2} \Delta s - \frac{4s}{t^3} \Delta t$$

$$= \frac{2}{t^2} \cdot \frac{1}{100} s - \frac{4s}{t^3} \cdot \frac{1}{400} t$$

$$= \frac{2s - s}{100 t^2}$$

$$= \frac{s}{100 t^2}$$

$g_0 = \frac{2s}{t^2}$

hampiran = $g_0 + \Delta G$

$$= \frac{2s}{t^2} + \frac{s}{100t^2}$$

$$= \frac{100(2s) + s}{100t^2}$$

$$= \frac{201(s)}{100t^2}$$

$$= \frac{402 s}{200 t^2}$$

$$= \frac{2s}{t^2} \cdot \frac{201}{200}$$

$$= \frac{201}{200} g \;//$$

7. Dapatkan $\frac{dz}{dt}$, jika :

b). $z = x\cos y + y\sin x$; $x = \sin 2t$; $y = \cos 2t$

$\frac{dx}{dt} = 2\cos 2t = 2y$; $\frac{dy}{dt} = -2\sin 2t = -2x$

$$dZ = \frac{\partial Z}{\partial x} dx + \frac{\partial Z}{\partial y} dy$$

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}$$

$$= (\cos y + y\cos x) \frac{dx}{dt} + [(-x\sin y) + \sin x] \frac{dy}{dt}$$

$$= (\cos y + y\cos x)\, 2y + [\sin x - x\sin y](-2x)$$

$$= 2y\cos y + 2y^2\cos x + 2x^2\sin y - 2x\sin x$$

$$= 2x^2\sin y + 2y\cos y + 2y^2\cos x - 2x\sin x$$

$$\frac{dz}{dt} = 2\left[x^2\sin y + y\cos y + y^2\cos x - x\sin x\right] \;//$$

8. Dapatkan $\frac{\partial z}{\partial s}$ dan $\frac{\partial z}{\partial t}$, jika :

b. $z = \frac{x}{y}$, $x = se^{t}$, $y = 1 + se^{-t}$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{1}{y}\cdot\frac{\partial x}{\partial s} + \frac{(-1)x}{y^2}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{1}{y}\cdot(e^{t}) - \frac{x}{y^2}\cdot(e^{-t})$$

$$= \frac{e^t}{1+se^{-t}} - \frac{se^t(e^{-t})}{(1+se^{-t})^2}$$

$$= \frac{e^t(1+se^{-t}) - s}{(1+se^{-t})^2}$$

$$= \frac{e^t + s - s}{(1+se^{-t})^2}$$

$$= \frac{e^t}{(1+se^{-t})^2} //$$

$$dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \frac{1}{y}(se^t) + \frac{(-x)}{y^2}\cdot(-se^{-t})$$

$$= \frac{s.e^t}{y} + \frac{x}{y^2}\cdot se^{-t}$$

$$= \frac{s.e^t}{(1+se^{-t})} + \frac{se^t(se^{-t})}{(1+se^{-t})^2}$$

$$= \frac{s.e^t(1+se^{-t}) + s^2}{(1+se^{-t})^2}$$

$$= \frac{se^t + s^2 + s^2}{(1+se^{-t})^2}$$

$$= \frac{2s^2 + se^t}{(1+se^{-t})^2} //$$

9. Jika $U = x^3y$ sedangkan x dan y fungsi dari t yang diberikan dalam bentuk :

$$x^5 + y = t$$
$$x^2 + y^3 = t^2$$

Tentukan $\frac{dU}{dt}$

$$dU = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= 3x^2y\,dx + x^3\,dy$$

$$\frac{dU}{dt} = 3x^2y\frac{dx}{dt} + x^3\frac{dy}{dt}$$

* $x^5 + y = t$

$$dt = \frac{\partial t}{\partial x}dx + \frac{\partial t}{\partial y}dy$$

$$\frac{dt}{dt} = 5x^4\frac{dx}{dt} + \frac{dy}{dt}$$

$$1 = 5x^4\frac{dx}{dt} + \frac{dy}{dt}$$

$$\frac{dy}{dt} = 1 - 5x^4\frac{dx}{dt} \quad \text{...(1)}$$

* $x^2 + y^3 = t^2$

misal : $F(x,y,t) = x^2 + y^3 - t^2 = 0$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt$$

$$\frac{dF}{dt} = 2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} - 2t\frac{dt}{dt} = 0$$

$$2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} - 2t = 0$$

$$2x\frac{dx}{dt} + 3y^2\left[1 - 5x^4\frac{dx}{dt}\right] = 2t$$

$$2x\frac{dx}{dt} + 3y^2 - 15x^4y^2\frac{dx}{dt} = 2t$$

$$(2x - 15x^4y^2)\frac{dx}{dt} = 2t - 3y^2$$

$$\frac{dx}{dt} = \frac{2t - 3y^2}{2x - 15x^4y^2}$$

$$\frac{dU}{dt} = 3x^2y\frac{dx}{dt} + x^3\frac{dy}{dt}$$

$$\rightarrow 3x^2y\left[\frac{2t-3y^2}{2x-15x^4y^2}\right] + x^3\left(1 - 5x^4\left[\frac{2t-3y^2}{2x-15x^4y^2}\right]\right) //$$

10 a. $w = x \cdot e^{y/z}$, $x = t^2$, $y = s - t$, $z = s + 2t$, maka $\frac{\partial w}{\partial s}$, $\frac{\partial w}{\partial t}$

dengan $s = 1$, $t = 0$

- $\frac{\partial x}{\partial s} = 0$ • $\frac{\partial y}{\partial s} = 1$ • $\frac{\partial z}{\partial s} = 1$ • $\frac{\partial x}{\partial t} = 2t$ • $\frac{\partial y}{\partial t} = -1$ • $\frac{\partial z}{\partial t} = 2$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$= e^{y/z} dx + \frac{x}{z} e^{y/z} dy + (-y) \times e^{y/z} dz$$

$$\frac{\partial w}{\partial s} = e^{y/z} \cdot \frac{\partial x}{\partial s} + \frac{x e^{y/z}}{z} \frac{\partial y}{\partial s} - xy e^{y/z} \frac{\partial z}{\partial s}$$

$$= e^{y/z} \cdot (0) + \frac{x}{z} e^{y/z} (1) - x \cdot y e^{y/z} \cdot (1)$$

$$= 0 + \frac{t^2}{s+2t} e^{\frac{s-t}{s+2t}} - t^2 (s-t) \cdot e^{s-t/s+2t}$$

$$= 0 + \frac{(0)^2}{1 + 2(0)} e - (0)^2 (1 - 0) \cdot e$$

$$= 0 + 0 - 0$$

$$= 0 //$$

$$\frac{\partial w}{\partial t} = e^{y/z} \cdot \frac{\partial x}{\partial t} + \frac{x}{z} e^{y/z} \cdot \frac{\partial y}{\partial t} - x \cdot y e^{y/z} \cdot \frac{\partial z}{\partial t}$$

$$= e^{\left(\frac{s-t}{s+2t}\right)} \cdot 2t + \frac{t^2}{s+2t} \cdot e^{s-t/s+2t} \cdot (-1) - t^2 (s-t) e^{s-t/s+2t} \cdot 2$$

$$= e \cdot 2(0) - \frac{0^2}{1+2(0)} e - 2 (0^2)(1-0) e$$

$$= 0 - 0 - 0$$

$$= 0 //$$

b). $w = xy + yz + xz$; $x = st$; $y = e^{st}$ $z = t^2$ maka $\frac{\partial w}{\partial s}$, $\frac{\partial w}{\partial t}$

$s = 0$ $t = 1$

- $\frac{\partial x}{\partial s} = t$ • $\frac{\partial y}{\partial s} = te^{st}$ • $\frac{\partial z}{\partial s} = 0$ • $\frac{\partial x}{\partial t} = s$ • $\frac{\partial y}{\partial t} = se^{st}$ • $\frac{\partial z}{\partial t} = 2t$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$= (y + z) dx + (x + z) dy + (y + x) dz$$

$$dW = (y+z)\,dx + (x+z)\,dy + (y+x)\,dz$$

$$\frac{\partial w}{\partial s} = (y+z)\frac{\partial x}{\partial s} + (x+z)\frac{\partial y}{\partial s} + (y+x)\frac{\partial z}{\partial s}$$

$$= (y+z)\,t + (x+z)\,t.e^{st} + (y+x)(0)$$

$$= (e^{st}+t^2)\,t + (st+t^2)\,te^{st} + 0$$

$$= (e^0+(1)^2)\,1 + (0(1)+(1)^2)\,(1).e^0$$

$$= (2)\,1 + (1)(1)(1)$$

$$= 2+1$$

$$= 3$$

$$\frac{\partial w}{\partial t} = (y+z)\frac{\partial x}{\partial t} + (x+z)\frac{\partial y}{\partial t} + (y+z)\frac{\partial z}{\partial t}$$

$$= (e^{st}+t^2).s + (st+t^2).se^{st} + (e^{st}+st)\,2t$$

$$= (e^0+(1)^2)\,0 + ((0)(1)+(1)^2)\,0.e^0 + (e^0+0).2(1)$$

$$= 0 + 0 + (1)(2)$$

$$= 2$$

11. a). Dapatkan $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, jika : $x^2 - xy + y^3 = 8$

$$F(x,y) : x^2 - xy + y^3 = 8$$

$$F(x,y) : x^2 - xy + y^3 - 8 = 0$$

$$dF : \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

$$: (2x-y)\,dx + (-x+3y^2)\,dy = 0$$

$$: (2x-y)\,dx = (x-3y^2)\,dy$$

$$\frac{dy}{dx} = \frac{2x-y}{x-3y^2} = \frac{y-2x}{3y^2-x}$$

$$G(x,y) : (-x+3y^2)\frac{dy}{dx} - y + 2x = 0$$

$$dG : \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = 0$$

$$\frac{\partial G}{\partial x} : \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0$$

$$\left[(-1)\frac{dy}{dx} + (-x+3y^2).\frac{d^2y}{dx^2} + 2\right] + \left[6y.\frac{dy}{dx} + (-x+3y^2)(0) - 1\right]\frac{dy}{dx} = 0$$

$$(-x+3y^2)\frac{d^2y}{dx^2} = -6y\left[\frac{dy}{dx}\right]^2 + \frac{dy}{dx} + \frac{dy}{dx} - 2$$

$$\frac{d^2y}{dx^2} = \frac{-6y\left[\frac{2x-y}{x-3y^2}\right]^2 + 2\left[\frac{2x-y}{x-3y^2}\right] - 2}{-x+3y^2}$$

$$= \frac{-6y(2x-y)(2x-y) + 2(2x-y)(x-3y^2) - 2(x-3y^2)(x-3y^2)}{(-x+3y^2)^3}$$

$$= \frac{-6y(4x^2-4xy+y^2) + 4x^2 - 12xy^2 - 2xy + 6y^3 - 2x^2 + 12xy^2 - 6y^4}{(-x+3y^2)^3}$$

$$= \frac{-24x^2y + 24xy^2 \cancel{-6y^3} \cancel{+6y^3} + 4x^2 - 2x^2 \cancel{-12xy^2} \cancel{+12xy^2} - 2xy + 6y^3 - 6y^4}{(-x+3y^2)^3}$$

$$= \frac{2x^2 - 6y^4 + 6y^3 - 2xy - 24x^2y + 24xy^2}{(-x+3y^2)^3}$$ //

11. b). Dapatkan $\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, jika $xyz = \cos(x+y+z)$

$F(x,y,z) : xyz - \cos(x+y+z) = 0$

$$dF : \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

$$[yz + \sin(x+y+z)]\,dx + [xz + \sin(x+y+z)]\,dy + [xy + \sin(x+y+z)]\,dz = 0$$

$$dz = \frac{-[yz + \sin(x+y+z)]\,dx}{[xy + \sin(x+y+z)]} - \frac{[xz + \sin(x+y+z)]}{[xy + \sin(x+y+z)]}dy$$

$$\frac{\partial z}{\partial x} = -\frac{[yz + \sin(x+y+z)]}{[xy + \sin(x+y+z)]} - \frac{[xz + \sin(x+y+z)]}{[xy + \sin(x+y+z)]}\frac{dy}{dx}$$ //

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x} = -\left[\frac{\cos(x+y+z)[xy + \sin(x+y+z)] - [yz + \sin(x+y+z)](y + \cos(x+y+z))}{[xy + \sin(x+y+z)^2]}\right]$$

$$- \left[\frac{z + \cos(x+y+z).[xy + \sin(x+y+z)] - [xz + \sin(x+y+z)][y + \cos(x+y+z)]}{[xy + \sin(x+y+z)]^2}\frac{dy}{dx} + \frac{[xz + \sin(x+y+z)]}{[xy + \sin(x+y+z)]}\cdot\frac{d^2y}{dx^2}\right]$$ //

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial\left(\frac{dz}{dx}\right)}{\partial y} = -\left[\frac{(z + \cos(x+y+z))(xy + \sin(x+y+z)) - (yz + \sin(x+y+z))[x + \cos(x+y+z)]}{[xy + \sin(x+y+z)]^2}\right]$$

$$- \left[\frac{\cos(x+y+z)[xy + \sin(x+y+z)] - [xz + \sin(x+y+z)](x + \cos(x+y+z))}{[xy + \sin(x+y+z)]^2}\cdot\frac{dy}{dx} + 0\right]$$ //

12. Jika $u = f(x-y, y-x)$, Tunjukkan $U_x + U_y = 0$

misal : $k = x-y$, maka $h = y-x$

$$du = \frac{\partial u}{\partial k} dk + \frac{\partial u}{\partial h} dh$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial k} \cdot \frac{\partial k}{\partial x} + \frac{\partial u}{\partial h} \cdot \frac{\partial h}{\partial x}$$

$$= \frac{\partial u}{\partial k}(1) + \frac{\partial u}{\partial h}(-1) \quad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial k} \cdot \frac{\partial k}{\partial y} + \frac{\partial u}{\partial h} \cdot \frac{\partial h}{\partial y}$$

$$= \frac{\partial u}{\partial k} \cdot (-1) + \frac{\partial u}{\partial h} \cdot (1) \quad \ldots (2)$$

$$U_x + U_y = \left[\frac{\partial u}{\partial k} - \frac{\partial u}{\partial h}\right] + \left[-\frac{\partial u}{\partial k} + \frac{\partial u}{\partial h}\right]$$

$$U_x + U_y = 0$$

// [TERBUKTI]

13. Jika $u = f(x,y)$ dan $x = r\cos\theta$, $y = r\sin\theta$. Tunjukkan bahwa :

$$(U_x)^2 + (U_y)^2 = (U_r)^2 + \frac{1}{r^2}(U_\theta)^2$$

$$dU = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$U_x = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$U_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}$$

$$U_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \sin\theta$$

$$U_\theta = \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} \cdot r\cos\theta$$

$$U_r^2 = \left(\frac{\partial u}{\partial x}\right)^2 \cos^2\theta + 2 \cdot \cos\theta\sin\theta \left[\frac{\partial^2 u}{\partial x \partial y}\right] + \left(\frac{\partial u}{\partial y}\right)^2 \sin^2\theta$$

$$\frac{U_\theta^2}{r^2} = \frac{\left(\frac{\partial u}{\partial x}\right)^2 r^2\sin^2\theta + (2)(-r^2)\cos\theta\sin\theta\left(\frac{\partial^2 u}{\partial x \partial y}\right) + \left(\frac{\partial u}{\partial y}\right)^2 r^2\cos^2\theta}{r^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 \sin^2\theta - 2\cos\theta\sin\theta\left(\frac{\partial^2 u}{\partial x \partial y}\right) + \left(\frac{\partial u}{\partial y}\right)^2 \cos^2\theta$$

$$U_r^2 + \frac{1}{r^2}(U_\theta)^2 = \left(\frac{\partial u}{\partial x}\right)^2\left[\cos^2\theta + \sin^2\theta\right] + 2\cos\theta\sin\theta\left[\frac{\partial^2 u}{\partial x \partial y}\right](1-1) + \left(\frac{\partial u}{\partial y}\right)^2\left[\sin^2\theta + \cos^2\theta\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + 0 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$= U_x + U_y$$

// [Terbukti]